

Many-particle entanglement with Bose–Einstein condensates

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We propose a method to produce entangled states of several particles starting from a Bose–Einstein condensate. In the proposal, a single fast $\pi/2$ pulse is applied to the atoms and due to the collisional interaction, the subsequent free time evolution creates an entangled state involving all atoms in the condensate. The created entangled state is a spin-squeezed state which could be used to improve the sensitivity of atomic clocks.

The possibility of creating and manipulating entangled states of many-particle systems has recently boosted the field of quantum information since it may yield new applications which rely on the basic principles of Quantum Mechanics¹. So far, up to four atoms have been entangled in a controlled way^{2,3}. The experimental achievement of Bose–Einstein condensation^{4–6} has also raised a lot of attention since it may lead to applications in several fields of Science. Some of these applications are based on the fact that condensates can be considered as pure states at the single particle level, which is a crucial requirement for the production of entangled state. Thus, a natural question is to investigate whether Bose–Einstein condensates can also be used in some applications of quantum information. In this paper we show that this is indeed the case. We propose a method to obtain substantial entanglement of a large number of atoms with present technology. In our proposal, a single resonant pulse is applied to all atoms in the condensate, and the collisional interaction entangles the atoms in the subsequent free evolution.

Consider a set of N two-level atoms confined by some external trap. In order to describe the internal properties of these atoms, it is convenient to let the internal states $|a\rangle_n$ and $|b\rangle_n$ of the n -th atom represent the two states of a fictitious spin 1/2 particle with angular momentum operators $j_z^{(n)} = 1/2(|a\rangle\langle a|_n - |b\rangle\langle b|_n)$, $j_x^{(n)} = 1/2(|b\rangle\langle a|_n + |a\rangle\langle b|_n)$, and $j_y^{(n)} = i/2(|b\rangle\langle a|_n - |a\rangle\langle b|_n)$. We consider collective effects of the atoms which are described by total angular momentum operators, $\vec{J} = \sum_{n=1}^N \vec{j}^{(n)}$. The entanglement properties of the atoms can be expressed in terms of the variances and expectation values of these operators. In the appendix we show that if

$$\xi^2 \equiv \frac{N(\Delta J_{\vec{n}_1})^2}{\langle J_{\vec{n}_2} \rangle^2 + \langle J_{\vec{n}_3} \rangle^2} < 1, \quad (1)$$

where $J_{\vec{n}} \equiv \vec{n} \cdot \vec{J}$ and the \vec{n} 's are mutually orthogonal unit vectors, then the state of the atoms is non–separable (i.e. entangled). The parameter ξ^2 thus characterizes the atomic entanglement, and states with $\xi^2 < 1$ are often referred to as “spin squeezed states”⁷. In the following we show how to reduce ξ^2 by several orders of magnitude using the collisional interactions between atoms in a Bose–Einstein condensate.

We consider a two component weakly interacting

Bose–Einstein condensate, which is present in several laboratories^{8,9}, and we assume that the interactions do not change the internal state of the atoms. This situation is described by the second quantized Hamiltonian

$$H = \sum_{j=a,b} \int d^3r \hat{\Psi}_j^\dagger(\mathbf{r}) H_{0,j} \hat{\Psi}_j(\mathbf{r}) + \frac{1}{2} \sum_{j=a,b} U_{jj} \int d^3r \hat{\Psi}_j^\dagger(\mathbf{r}) \hat{\Psi}_j^\dagger(\mathbf{r}) \hat{\Psi}_j(\mathbf{r}) \hat{\Psi}_j(\mathbf{r}) + U_{ab} \int d^3r \hat{\Psi}_a^\dagger(\mathbf{r}) \hat{\Psi}_b^\dagger(\mathbf{r}) \hat{\Psi}_a(\mathbf{r}) \hat{\Psi}_b(\mathbf{r}), \quad (2)$$

where $H_{0,j}$ is the one particle Hamiltonian for atoms in state j including the kinetic energy and the external trapping potential $V_j(\mathbf{r})$, $\hat{\Psi}_j(\mathbf{r})$ is the field operator for atoms in the state j , $U_{jk} = 4\pi\hbar^2 a_{jk}/m$ is the strength of the interaction between particles of type j and k , parameterized by the scattering length a_{jk} , and m is the atomic mass.

Assume that we start with a Bose–Einstein condensate in state $|a\rangle$ at very low temperature ($T \simeq 0$), so that all the atoms are in a single particle (motional) state $|\phi_0\rangle$. A fast $\pi/2$ pulse between the states $|a\rangle$ and $|b\rangle$ prepares the atoms in the state $|\phi_0\rangle^{\otimes N} \otimes (|a\rangle + |b\rangle)^{\otimes N}/2^{N/2}$ which is an eigenstate of the J_x operator with eigenvalue $N/2$. If we choose in (1) $\vec{n}_1 = [0, \cos(\theta), \sin(\theta)]$ and \vec{n}_2 along the x axis of the fictitious angular momentum, we have $\xi_\theta^2 = 1$ at $t = 0$. By using the equations of motion of the angular momentum operators in the Heisenberg picture we find the time derivative of ξ_θ^2 at $t = 0$

$$\frac{d}{dt} \xi_\theta^2 = \sin(2\theta) \frac{(N-1)(U_{aa} + U_{bb} - 2U_{ab})}{2\hbar} \int d^3r |\phi_0|^4. \quad (3)$$

This equation immediately shows that spin squeezing will be produced for certain angles θ if $U_{aa} + U_{bb} \neq 2U_{ab}$ as it is in the Na experiments⁹.

To quantify the amount of squeezing which may be obtained, we assume identical trapping potentials $V_a(\mathbf{r}) = V_b(\mathbf{r})$ and identical coupling constants for interaction between atoms in the same internal state $a_{aa} = a_{bb} > a_{ab}$. Physically, this could correspond to the $|F = 1, M_F = \pm 1\rangle$ hyperfine states of Na trapped in an optical dipole trap. Due to the symmetry of these states their scattering lengths and trapping potentials will be identical; moreover, due to angular momentum conservation there are no spin exchanging collisions between these states as required by our model. This is exactly the experimental setup used in Ref. 9. In this experiment it is shown that these states have an antiferromagnetic

interaction $a_{ab} < a_{aa}$ which according to (3) enables the production of squeezed states. To avoid spin changing collisions that populate the state $|M_F = 0\rangle$ ¹⁰ one has to slightly modify such an experimental set-up. If the $F = 1$ manifold is coupled to the $F = 2$ with a far off-resonant blue detuned π -polarized microwave field, the $|F = 1, M_F = 0\rangle$ state is raised in energy with respect to the $|F = 1, M_F = \pm 1\rangle$ states, since the Clebsch-Gordan coefficient for the $|F = 1, M_F = 0\rangle \rightarrow |F = 2, M_F = 0\rangle$ is larger than the coefficients for the $|F = 1, M_F = \pm 1\rangle \rightarrow |F = 2, M_F = \pm 1\rangle$ transitions, and spin exchanging collisions become energetically forbidden. If for instance one chooses a detuning $\delta = (2\pi)25$ MHz and a resonant Rabi frequency for the $1 \rightarrow 1$ transition of $\Omega = (2\pi)2$ MHz, the energy separation is $\Delta E = 640$ nK. With a typical chemical potential $\mu \approx 220$ nK¹⁰, this energy separation is much higher than the available energy in the collisions and the $|F = 1, M = 0\rangle$ state is completely decoupled. In Ref. 10 a much smaller energy difference is shown to exclude spin exchanging collision and we therefore expect that much weaker fields will suffice.

The assumption $a_{aa} = a_{bb}$ has several advantages. Firstly, it reduces the effect of fluctuations in the total particle number. If $a_{aa} \neq a_{bb}$ the mean spin performs a N dependent rotation around the z -axis, and fluctuations in the number of particles introduces an uncertainty in the direction of the spin which effectively reduces the average value and introduces noise into the system. With $a_{aa} = a_{bb}$ the mean spin remains in the x -direction independent of the number of atoms in the trap. Secondly, this condition ensures a large spatial overlap of different components of the wavefunction. After the $\pi/2$ pulse the spatial wavefunction is no longer in the equilibrium state. That is, due to the atomic repulsions (which are now different than before since the atoms are in different internal states), the spatial distribution of the atomic cloud will start oscillating. Furthermore, since the state of the system is now distributed over a range of number of particles in the $|a\rangle$ state (N_a), and since this number enters into the time evolution, the N_a dependent wavefunctions ϕ_a and ϕ_b are different for particles in the states $|a\rangle$ and $|b\rangle$. With $a_{aa} = a_{bb}$, ϕ_a and ϕ_b are identical if N_a equals the average number $N/2$. In the limit of large N , the width of the distribution on different N_a 's is much smaller than N_a and all the spatial wavefunctions are approximately identical $\phi_a(N_a, t) \approx \phi_b(N_a, t) \approx \phi_0(t)$. This relation is only true if $a_{aa} > a_{ab}$ where small deviations from the average wavefunction perform small oscillations. In the opposite case the deviations grow exponentially¹¹ resulting in a reduction of the overlap of the a and the b wavefunctions and a reduced squeezing. The advantages mentioned above could also be achieved with $a_{aa} \neq a_{bb}$ by using the breath-together solutions proposed in Ref. 11.

Before analyzing quantitatively the complete system, we estimate the amount of spin squeezing we can reach with our proposal by using a simple model. Assuming the same wavefunction ϕ_0 for both $|a\rangle$ and $|b\rangle$ atoms is constant and independent of the number N_a , the spin dependent part of the Hamiltonian (2) may be written as $H_{\text{spin}} = \hbar\chi J_z^2$, where χ depends on the scattering lengths and the wavefunction ϕ_0 . The spin squeezing produced by this Hamiltonian can be cal-

culated exactly⁷. In the limit of large N , the minimum obtainable squeezing parameter is $\xi_\theta^2 = \frac{1}{2} \left(\frac{3}{N}\right)^{2/3}$, which indicates that our proposal might produce a reduction of ξ^2 by a factor of $\sim N^{2/3}$ which would be more than three orders of magnitude with 10^5 atoms in the condensate.

In contrast to the simplified Hamiltonian H_{spin} the real Hamiltonian (2) will also entangle the internal and motional states of the atoms which is a source of decoherence for the spin squeezing. To quantify this effect we have performed a direct numerical integration following the procedure developed in Ref. 11. We split the whole Hilbert space into orthogonal subspaces containing a fixed number of particles N_a and $N_b = N - N_a$ in each of the internal states, respectively. In each subspace we make a Hartree-Fock variational ansatz in terms of three-dimensional spatial wavefunctions $\phi_a(N_a, t)$ and $\phi_b(N_b, t)$, which are evolved according to the time dependent coupled Gross-Pitaevskii equations. This is an approximation to the full problem which is valid in the limit of low temperatures and short times, where the population of the Bogoliubov modes is small. Particularly it is a good approximation in our case with $a_{ab} < a_{aa}, a_{bb}$ where there are no demixing instabilities. With this procedure, the decoherence induced by the entanglement with the motional state is effectively taken into account. Together with the prediction from the simple Hamiltonian H_{spin} the result of the simulation is shown in Fig. 1. The two curves are roughly in agreement confirming that the system is able to approximate the results of the Hamiltonian H_{spin} . The numerical solution shows fluctuations due to the oscillations of the spatial wavefunction. The large dips at $\omega t \approx 4, 9, 13$, and 18 are the points where the atomic cloud reach the initial width. At these instants the overlap of the wavefunctions is maximal and the two curves are very close (up to a factor of two). The small dips at $\omega t = 2, 7, 11$, and 16 corresponds to the points of maximum compression. With the realistic parameters used in the figure, our simulation suggests that three orders of magnitude squeezing is possible. Also, note the time scale in the figure. The maximally squeezed state is reached after approximately two oscillation periods in the trap. For a fixed ratio of the scattering lengths a_{ab}/a_{aa} , the optimal time scales as $(a_{aa}/d_0)^{-2/5} N^{-1/15}$, where $d_0 = \sqrt{\hbar/(m\omega)}$ is the width of the ground state of the harmonic potential.

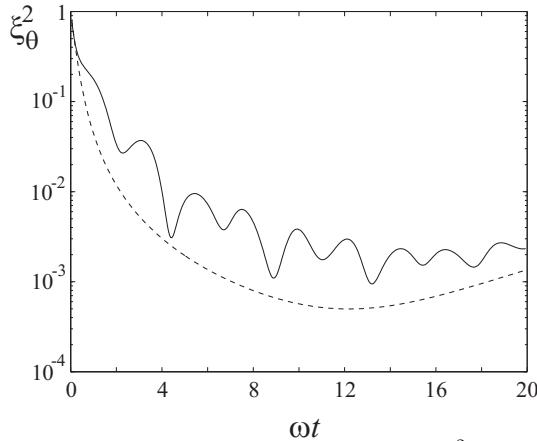


FIG. 1. Reduction in the squeezing parameter ξ^2 . A fast $\pi/2$ pulse between two internal states is applied to all atoms in the condensate. The subsequent free time evolution results in a strong squeezing of the total spin. The angle θ is chosen such that ξ_θ^2 is minimal. The solid line is the results of a numerical integration (see text). For numerical convenience we have assumed a spherically symmetric potential $V(r) = m\omega^2 r^2/2$. The parameters are $a_{aa}/d_0 = 6 * 10^{-3}$, $a_{bb} = 2a_{ab} = a_{aa}$, and $N = 10^5$. The dashed curve shows the squeezing obtained from the Hamiltonian $H_{\text{spin}} = \hbar\chi J_z^2$. The parameter χ is chosen such that the reduction of J_x obtained from the solution in Ref. 7 is consistent with the results of Ref. 11.

The analysis so far has left out a number of possible imperfections. Specifically, we have assumed that all scattering length are real so that no atoms are lost from the trap and we have not considered the role of thermal particles. To estimate the effect particle losses we have performed a Monte Carlo simulation¹² of the evolution of squeezing from the Hamiltonian H_{spin} . The particle loss is phenomenologically taken into account by introducing a loss rate Γ which is identical for atoms in the $|a\rangle$ and the $|b\rangle$ state. In Fig. 2 we show the obtainable squeezing in the presence of loss. Approximately 10 % of the atoms are lost at the time $\chi t \approx 6 * 10^{-4}$ where the squeezing is maximally without loss. With the parameters of Fig. 1 this time corresponds to roughly two trapping periods. Such a large loss is an exaggeration of the loss compared to current experiments and the simulation indicate that even under these conditions, squeezing of nearly two orders of magnitude may be obtained. On the other hand, the effects of thermal particles can be suppressed at sufficiently low temperatures but due to the robustness with respect to particle losses shown in Fig. 2, we expect to obtain high squeezing even at some finite temperatures.

In conclusion we believe that we have presented a simple and robust method to produce entangled states of a large number of atoms with present technology. The produced entangled state are interesting in fundamental physics and they also have possible technological applications in atomic clocks^{13,14}, where the projection noise $(\Delta J_\theta)^2$ is currently the main source of noise¹⁵. In particular Boyer and Kasevich¹⁴ have shown how to use an entangled state with half of the particles $N/2$ in one internal state a , and the other

half in the other state b to obtain the Heisenberg limit in atomic interferometric measurements. On the other hand, it has been shown that if the atoms are prepared in a state with $\xi^2 < 1$ before they are injected into an atomic clock, one can reduce the frequency noise (variance in the frequency measurements) or the measuring time to obtain a desired precision by a factor ξ^2 as compared to the case in which one uses atoms in an uncorrelated state¹³. If the squeezing produced by our proposal is transferred into a suitable clock transition, and the atomic cloud is allowed to expand so that the role of collisions is reduced during the frequency measurement, the states could be directly applied in the current set-up of atomic clocks. Other theoretical proposals for noise reduction in neutral atom systems have been made¹⁶, and a weak squeezing of the spin has recently been produced experimentally^{17,18}. However, our proposal has the considerable advantage that it offers a very strong noise reduction, and it is directly applicable to existing experimental set-ups. In future experiments with negligible particle loss even for very long interaction times, the Hamiltonian H_{spin} could also be used to produce maximally entangled state of any number of atoms^{2,19}, which could reduce the frequency noise to the fundamental limit of quantum mechanics²⁰.

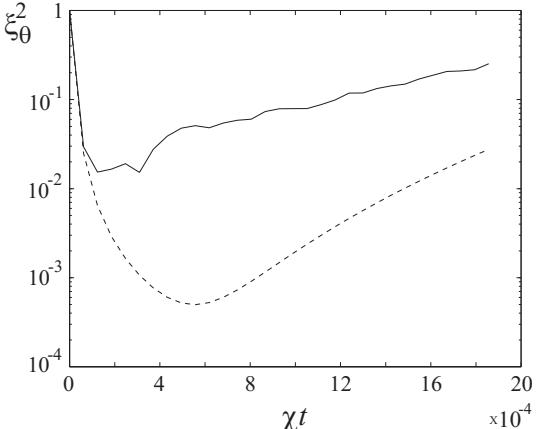


FIG. 2. Quantum Monte Carlo simulation of squeezing in the presence of loss. The full line shows squeezing obtained from the Hamiltonian H_{spin} with particle loss described by a constant loss rate $\Gamma = 200\chi$. The dashed curve shows squeezing without particle loss.

Appendix

Here we present the derivation of Eq. (1) as a criterion for entanglement. An N -particle density matrix ρ is defined to be separable (non-entangled) if it can be decomposed into

$$\rho = \sum_k P_k \rho_k^{(1)} \otimes \rho_k^{(2)} \otimes \dots \otimes \rho_k^{(N)}, \quad (4)$$

where the coefficients P_k are positive real numbers fulfilling $\sum_k P_k = 1$, and $\rho_k^{(i)}$ is a density matrix for the i 'th particle. The variance of J_z may be described as $(\Delta J_z)^2 = \frac{N}{4} - \sum_k P_k \sum_i \langle j_z^{(i)} \rangle_k^2 + \sum_k P_k \langle J_z \rangle_k^2 - \langle J_z \rangle^2$, and using Cauchy-Schwarz's inequality and $\langle j_x^{(i)} \rangle_k^2 + \langle j_y^{(i)} \rangle_k^2 + \langle j_z^{(i)} \rangle_k^2 \leq 1/4$

we find three inequalities for separable states $\sum_k P_k \langle J_z \rangle_k^2 \geq \langle J_z \rangle^2$, $-\sum_k P_k \sum_i \langle j_z^{(i)} \rangle_k^2 \geq -\frac{N}{4} + \sum_k P_k \sum_i \langle j_x^{(i)} \rangle_k^2 + \langle j_y^{(i)} \rangle_k^2$, and $\langle J_x \rangle^2 \leq N \sum_k P_k \sum_i \langle j_x^{(i)} \rangle_k^2$. From these inequalities we immediately find that any separable state obeys $\xi^2 \geq 1$ and hence any state with $\xi^2 < 1$ is non-separable.

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